

PAIRS OF MATRICES IN $GL_2(\mathbf{R}_{\geq 0})$ THAT FREELY GENERATE

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ABSTRACT. An elementary proof that certain pairs of 2×2 matrices with nonnegative real coordinates generate free monoids.

A monoid is a semigroup with an identity. Let $GL_2(\mathbf{R}_{\geq 0})$ denote the multiplicative monoid of 2×2 matrices with nonzero determinant and with coordinates in the set $\mathbf{R}_{\geq 0}$ of nonnegative real numbers. To every matrix

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} \in GL_2(\mathbf{R}_{\geq 0})$$

we associate the linear fractional transformation

$$X(t) = \frac{x_{1,1}t + x_{1,2}}{x_{2,1}t + x_{2,2}}$$

defined on the set of positive real numbers t . This is a monoid isomorphism from $GL_2(\mathbf{R}_{\geq 0})$ to the monoid of linear fractional transformations with nonnegative real coordinates, nonzero determinant, and the binary operation of composition of functions. Without loss (or gain) of generality, we can use the language of matrices or the language of functions.

It is important to note that the nonzero determinant of X and the nonnegativity of the coordinates of X imply that if $t > 0$, then $X(t) > 0$.

The monoid $M(A, B)$ generated by a pair of matrices $\{A, B\}$ in $GL_2(\mathbf{R}_{\geq 0})$ consists of all matrices that can be represented as products of nonnegative powers of A and B . The matrices A and B *freely generate* this monoid if every matrix in $M(A, B)$ has a unique representation as a product of powers of A and B .

Consider the matrices $L_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The Calkin-Wilf tree [1, 3] is the directed graph whose vertices are the positive rational numbers and which is constructed inductively from the root vertex 1 by the generation rule

$$\begin{array}{ccc} & t & \\ \swarrow & & \searrow \\ L_1(t) = \frac{t}{t+1} & & R_1(t) = t+1 \end{array}$$

The fact that the Calkin-Wilf graph is a tree is equivalent to the well-known folk theorem that the matrices L_1 and R_1 freely generate the monoid $SL_2(\mathbf{N}_0)$ of 2×2 matrices with determinant 1 and nonnegative integral coordinates.

It is a standard application of the ping-pong lemma that for every pair (u, v) of integers with $u \geq 2$ and $v \geq 2$, the matrices $L_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ and $R_v = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$

generate a free group of rank 2 (Lyndon and Schupp [2, pp. 167–168]). The case $u = v = 2$ is Sanov's theorem [4].

In this note we give a simple proof that certain pairs of matrices in $GL_2(\mathbf{R}_{\geq 0})$ freely generate a monoid. The examples above are special cases of this result.

Theorem 1. *Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ and $B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$ be matrices in $GL_2(\mathbf{R}_{\geq 0})$. If*

$$(1) \quad a_{1,1} \leq a_{2,1} \quad \text{and} \quad a_{1,2} \leq a_{2,2}$$

and if

$$(2) \quad b_{1,1} \geq b_{2,1} \quad \text{and} \quad b_{1,2} \geq b_{2,2}$$

then the submodule of $GL_2(\mathbf{R}_{\geq 0})$ generated by A and B is free, and $\{A, B\}$ is a free set of generators for this module.

Proof. We associate to the matrices A and B the linear fractional transformations

$$A(t) = \frac{a_{1,1}t + a_{1,2}}{a_{2,1}t + a_{2,2}} \quad \text{and} \quad B(t) = \frac{b_{1,1}t + b_{1,2}}{b_{2,1}t + b_{2,2}}.$$

Let $t > 0$. Because $\det(A) = a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0$, we have $a_{2,1}t + a_{2,2} > 0$. Inequalities (1) imply that

$$(a_{1,1} - a_{2,1})t \leq 0 \leq a_{2,2} - a_{1,2}.$$

If $(a_{1,1} - a_{2,1})t = a_{2,2} - a_{1,2}$, then $a_{1,1} = a_{2,1}$ and $a_{2,2} = a_{1,2}$, and so $\det(A) = 0$, which is absurd. Therefore, $(a_{1,1} - a_{2,1})t < a_{2,2} - a_{1,2}$ or, equivalently, $0 < A(t) < 1$. Similarly, inequalities (2) imply that if $t' > 0$, then $(b_{1,1} - b_{2,1})t' > b_{2,2} - b_{1,2}$ and so $B(t') > 1$. Thus, for all $t, t' > 0$, we have

$$(3) \quad 0 < A(t) < 1 < B(t').$$

If A and B do not freely generate a monoid, then there exist distinct sequences (X_1, X_2, \dots, X_k) and $(Y_1, Y_2, \dots, Y_\ell)$ with $X_i \in \{A, B\}$ for $i = 1, \dots, k$ and $Y_j \in \{A, B\}$ for $j = 1, \dots, \ell$ such that

$$(4) \quad X_1 X_2 \cdots X_k = Y_1 Y_2 \cdots Y_\ell.$$

Choose the smallest positive integer k for which a relation of the form (4) exists. If $X_1 = Y_1$, then $X_2 \cdots X_k = Y_2 \cdots Y_\ell$, which contradicts the minimality of k . Therefore, $X_1 \neq Y_1$.

Suppose that $X_1 = A$ and $Y_1 = B$. Applying the matrices as linear fractional transformations, we obtain $t = X_2 \cdots X_k(1) > 0$ and $t' = Y_2 \cdots Y_\ell(1) > 0$. Identity (4) implies that

$$A(t) = X_1 X_2 \cdots X_k(1) = Y_1 Y_2 \cdots Y_\ell(1) = B(t').$$

This is absurd because it contradicts inequality (3). The case $X_1 = B$ and $Y_1 = A$ is similar. This completes the proof. \square

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